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Emergence of on-off intermittency in systems nonlinearly coupled to a nonequilibrium bath

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Approximate analytic solutions are presented for the dynamics of a classical oscillator nonlinearly coupled to a nonequilibrium bath. It is shown that as a result of the combined effect of the nonlinear coupling, which leads to nonlinear friction and multiplicative noise in the description of the reduced system, and the nonthermal properties of the reservoir, which give a specific self-sustained character to the coarse-grained oscillator, on-off intermittency can occur. Properties of this phenomenon, such as the universality in the length distribution of the laminar phase and the qualitative changes caused by the presence of additive noise, can be traced back to characteristics of the starting microscopic model. [S1063-651X(99)01411-7]

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I. INTRODUCTION

''On-off intermittency'' denotes an intermittent behavior significantly different from the Pomeau-Manneville and crisis-induced types [1]. Opposed to the *static* character of the bifurcation parameter in those cases, in on-off intermittency it is the stochastic or chaotic parametric variation through the bifurcation point that gives rise to the aperiodic switching between the laminar (off) phase and the burst (on) phase. The phenomenon, which can be identified by specific properties such as the universal power-law distribution for the laminar phase and the particular dependence of the average length of this phase with the *coupling parameter*, has been reported to appear in different systems [2–6]. In this paper we present an analytical study of the emergence of on-off intermittency in the coarse-grained dynamics of systems nonlinearly coupled to a nonequilibrium reservoir. The interest in having a derivation from a microscopic model of a self-consistently reduced description of this effect is clear. From a fundamental point of view, it can establish a connection between the deterministic or stochastic phenomenological equations and the underlying Hamiltonian dynamics; consequently, a physical basis for some of the mathematical models can be found, and the fundamental origin of some of the mechanisms proposed as responsible for the effect can be traced. In more practical terms, the understanding of the emergent character of the phenomenon can help to identify diverse contexts in which it can be relevant.

The outline of the paper is as follows. In Sec. II, we present a ''microscopic'' model for the study of on-off intermittency, and, through a coarse-graining process, derive a reduced description of it in terms of a stochastic system. In Sec. III, analytical solutions for the emergent dynamics are obtained applying the averaging methods of Bogoliubov, Krylov, and Stratonovich. Subsequently, the signatures of the intermittent behavior are related to the characteristics of the initial Hamiltonian system. The subject of Sec. IV is the study, in the same framework, of the role of additive noise in altering some of the properties of the effect; our results are compared with the ones corresponding to standard models for ''noisy'' on-off intermittency. Finally, in Sec. V some conclusions are summarized.

II. THE MODEL

Our starting point is the microscopic model proposed in Ref. [7] to study nonexponential decay of correlation functions. Specifically, we consider the classical dynamics of a complex system which consists of a particular singled out, linear or nonlinear, mode (q,p), and a set of linear modes $\{(Q_k,P_k)\}$. These subsystems are coupled, nonlinearly in q [through the function $V(q)$] and linearly in Q_k; additionally, the mode (q,p) is bilinearly coupled to a thermal reservoir of harmonic oscillators B. The Hamilton function reads

$$H=\frac{1}{2}p^2+U(q)+\frac{1}{2}\sum_k (P_k^2+\Omega_k^2Q_k^2)-V(q)\sum_k Q_k+H_B+H_{\rm int}(B,q). \qquad (1)$$

Assuming the weak-coupling limit for all the interactions, an Ohmic character for the thermal bath, a Debye-type frequency density for the phonons $\{Q_k\}$ with a cutoff Ω_c much larger than any typical frequency of the mode q, and following standard techniques [8], it is shown that the coarse-grained dynamics for the mode q is described by [7]

$$\ddot{q}=-\Gamma(q)p-\tilde{U}'(q)+\xi_b(t)+V'(q)\xi_p(t), \qquad (2)$$

where $\Gamma(q)=\gamma_b+\gamma_p[V'(q)]^2$ gives the nonlinearity of the friction term (γ_b and γ_p are the dissipation coefficients due to the thermal bath and the phonons $\{Q_k\}$, respectively); $\tilde{U}(q)=U(q)-(\Omega_c/\pi)\gamma_pV^2(q)$ is the *bare* potential *dressed* by the modes $\{Q_k\}$ [the *dressing* caused by the thermal bath is absorbed in $U(q)$]; $\xi_b(t)$ is Gaussian white noise, i.e., $\langle\xi_b(t)\rangle=0$ and $\langle\xi_b(t)\xi_b(t')\rangle=2\gamma_bk_BT\delta(t-t')$; and $\xi_p(t)$ is zero-mean Gaussian noise (a random distribution for the initial phases of the phonons is assumed) with a spectral density [7]

$$S[\xi_p;\omega]=2\int_{-\infty}^{\infty}d\tau\langle\xi_p(t)\xi_p(t+\tau)\rangle e^{i\omega\tau}=8\gamma_pu(\omega), \qquad (3)$$

$u(\omega)$ being the energy density of the nonequilibrium state assumed for the modes $\{Q_k\}$.

We emphasize two main points in this picture: first, the coarse-graining performed in our starting Hamiltonian model has led to a reduced bidimensional system with nonlinear friction and both additive and multiplicative noise terms; and second, the spectral properties of $\xi_p(t)$ are determined by the state of the phonons.

III. THE AVERAGING METHOD

Now, let us see how our model system, which was studied in Ref. [7] in the overdamped limit, can be solved analytically in the underdamped case under certain restrictions. First, we consider that the potential $\tilde{U}(q)$ corresponds to a harmonic oscillator perturbed by nonlinear terms [for simplicity we assume that the nonlinear part comes only from the *dressing*, therefore $U(q)=(1/2)\omega_0^2 q^2$]. Second, if the friction term and the fluctuational forces can also be considered as a perturbation to the harmonic potential, we can apply the Bogoliubov-Krylov methods [9] to average the system [10]. In this sense, we choose as definitions for the amplitude A and phase $\psi=\omega_0 t+\varphi$ the equations $q=A\cos(\omega_0 t+\varphi)$ and $p=-\omega_0 A\sin(\omega_0 t+\varphi)$. With these changes Eq. (2) is converted into a system of two equations in *standard form* in which the averaging of the deterministic terms over $\tau_0=2\pi/\omega_0$ leads to

$$\begin{aligned}\dot{A}&=-\frac{\gamma_b}{2}A-\frac{\alpha^2\gamma_p}{8}A^3-\frac{1}{\omega_0}\xi_b(t)\sin(\omega_0 t+\varphi)\\&\quad-\frac{\alpha}{2\omega_0}A\xi_p(t)\sin(2\omega_0 t+2\varphi),\\\dot{\varphi}&=-\frac{3\alpha^2\gamma_p\Omega_c}{8\pi\omega_0}A^2-\frac{1}{\omega_0}\frac{1}{A}\xi_b(t)\cos(\omega_0 t+\varphi)\\&\quad-\frac{\alpha}{2\omega_0}\xi_p(t)[1+\cos(2\omega_0 t+2\varphi)],\end{aligned}\tag{4}$$

where we have taken for the nonlinear coupling $V(q)=(1/2)\alpha q^2$. The stochastic terms can also be averaged if they have sufficiently small correlation times. This is the case of $\xi_b(t)$, and also of $\xi_p(t)$ if the phonons $\{Q_k\}$ are in a state with a broadband spectral density. When these conditions are met, we can, following Ref. [10], approximate each of the noise terms of the previous equations by the sum of its double, statistical, and, over $\tau_0=2\pi/\omega_0$, average, and a zero-mean Gaussian white random force. In this way we have

$$\xi_b(t)\sin(\omega_0 t+\varphi)=\overline{\langle\xi_b(t)\sin(\omega_0 t+\varphi)\rangle}+\zeta_{b,1}(t),\tag{5}$$

where the bar indicates time averaging and the angular brackets mean averaging over the statistical ensemble. Using similar expressions for the other stochastic terms, it is shown that in first order the coarse-grained system is described by the Stratonovich equations

$$\begin{aligned}\dot{A}&=\left(-\frac{\gamma_b}{2}+\frac{\alpha^2\gamma_p u(2\omega_0)}{2\omega_0^2}\right)A-\frac{\alpha^2\gamma_p}{8}A^3+\frac{\gamma_b k_B T}{2\omega_0^2}\frac{1}{A}\\&\quad-\frac{1}{\omega_0}\zeta_{b,1}(t)-\frac{\alpha}{2\omega_0}A\zeta_{p,1}(t),\\\dot{\varphi}&=-\frac{3\alpha^2\gamma_p\Omega_c}{8\pi\omega_0}A^2+m-\frac{1}{\omega_0}\frac{1}{A}\zeta_{b,2}(t)-\frac{\alpha}{2\omega_0}\zeta_{p,2}(t),\end{aligned}\tag{6}$$

where $\zeta_{b,i}(t)$ and $\zeta_{p,i}(t)$ $(i=1,2)$ are zero-mean Gaussian white-noise terms, i.e., $\langle\zeta_{b,i}(t)\zeta_{b,j}(t')\rangle=\delta_{i,j}K_{b,i}\delta(t-t')$ and $\langle\zeta_{p,i}(t)\zeta_{p,j}(t')\rangle=\delta_{i,j}K_{p,i}\delta(t-t')$ with $K_{b,1}=K_{b,2}=\gamma_b k_B T$, $K_{p,1}=2\gamma_p u(2\omega_0)$, and $K_{p,2}=4\gamma_p[u(0)+(1/2)u(2\omega_0)]$; and the shift of the frequency due to the colored noise is

$$m=\frac{\alpha^2}{4\omega_0^2}\int_{-\infty}^{0}d\tau\langle\xi_p(0)\xi_p(\tau)\rangle\sin(2\omega_0\tau).$$

For $T=0$ we recognize in the equation for A the functional form of one of the stochastic models used to account for on-off intermittency [3,5,6]. As we can give analytic, both stationary and time-dependent, solutions for this equation, we can relate the signatures of the phenomenon with the properties of our starting Hamiltonian system. This is the subject of the rest of this section. The effects of a nonzero temperature will be studied in Sec. IV.

A. Stationary solutions

In the $T=0$ case, the stationary probability density for the amplitude is given by [10–13]

$$W_{SS}(A)=\begin{cases}\dfrac{2\Lambda^\nu}{\Gamma(\nu)}A^{2\nu-1}e^{-\Lambda A^2} & \text{for } \nu>0\\ \delta(A) & \text{for } \nu\leqslant 0,\end{cases}\tag{7}$$

with $\nu=1-\gamma_b\omega_0^2/[\alpha^2\gamma_p u(2\omega_0)]$ and $\Lambda=\omega_0^2/[4u(2\omega_0)]$, and it has a variance $\langle A^2\rangle-\langle A\rangle^2=[\nu-\Gamma^2(\nu+1/2)/\Gamma^2(\nu)]/\Lambda$.

Let us discuss the different qualitative behaviors that, depending on ν, exist in the system [10]. In the range $0<\nu<1/2$, the probability density is monotonic and the most probable A is zero: no ''predominant'' amplitude exists in this regime of ''undeveloped oscillations.'' For $1/2<\nu<1$, $W_{SS}(A)$ has the maximum at $A_m=[(2\nu-1)/(2\Lambda)]^{1/2}$; values of A near the origin are still quite probable, however. Finally, in the region of ''fully developed oscillations,'' $\nu>1$, the system remains mainly around a ''limit cycle.'' It can be advanced that it is in the first regime that intermittent behavior can occur; moreover, it can be rigorously shown that for $0<\nu\ll 1$, the system exhibits properties specific to on-off intermittency: a power-law density $W_{SS}(A)\sim A^{2\nu-1}$ for small A is obtained from Eq. (7), and a $-3/2$ power function for the length distribution of the laminar phase at the onset of the oscillations is found by mapping the process to a random walk [6]. In our model, the maximum value of ν is 1, and it is reached when q is disconnected from the thermal bath ($\gamma_b=0$). In that case, $W_{SS}(A)$ equals a Rayleigh distribution [10] and hence the system never enters the intermittency domain. A Rayleigh distribution is equally obtained if $\gamma_p=0$ (the mode q coupled only to the thermal bath). On the contrary, for $\gamma_b\neq 0$ and $\gamma_p\neq 0$, the reduced system, which is always outside the ''fully developed oscillations'' regime, shows intermittent behavior if the energy distribution $u(2\omega_0)$ is slightly larger than the threshold $\omega_0^2\gamma_b/(\alpha^2\gamma_p)$. In this regime, smaller values of ν result first in an increase of the *mean laminar phase*, which goes to infinity at the threshold, and finally in the quenching of the burst phase; additionally there is also an increase of Λ and

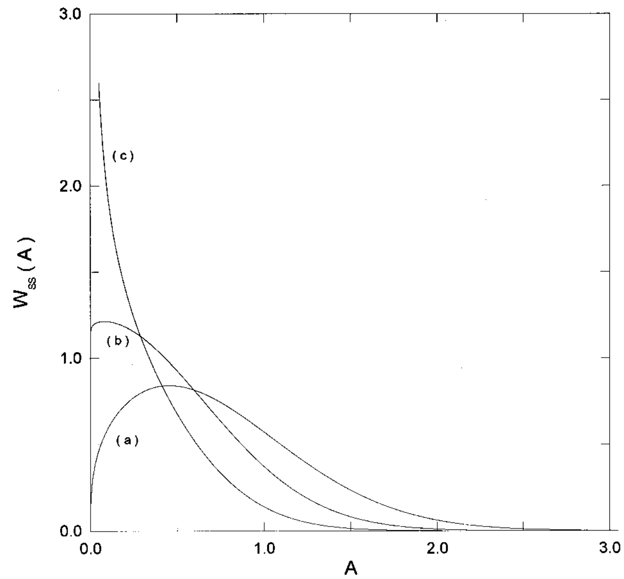

FIG. 1. Stationary probability density $W_{SS}(A)$ for $T=0$, $\gamma_b=0.01$, and $u(2\omega_0)=0.3$ (a), $u(2\omega_0)=0.2$ (b), and $u(2\omega_0)=0.143$ (c). The rest of the parameters of the system are $\gamma_p=0.1$, $\omega_0=1$, $\alpha=1$, and are the same in all the figures.

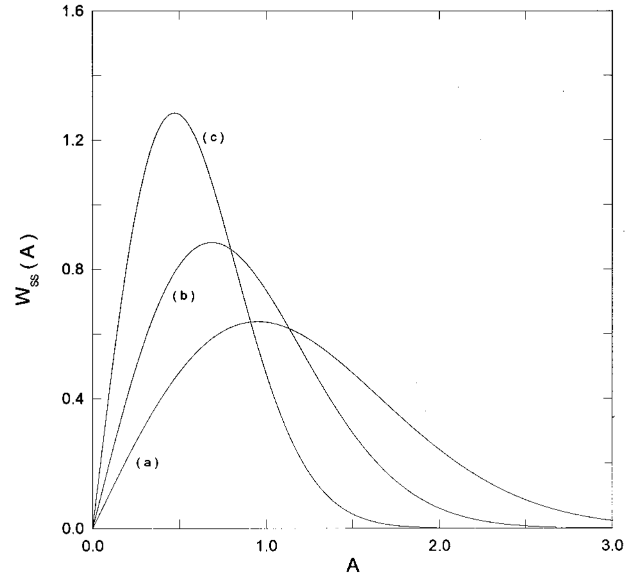

FIG. 2. Stationary probability density $W_{SS}(A)$ for $T=0$, $\gamma_b=0$ $(\nu=1)$, and $u(2\omega_0)=0.45$ (a), $u(2\omega_0)=0.236$ (b), and $u(2\omega_0)=0.111$ (c).

therefore a reduction of the effective width of the density. In Fig. 1, where $W_{SS}(A)$ is depicted for different values of ν, the dependence of the qualitative behavior of the system on the preparation of the phonons is clearly reflected. In contrast, in Fig. 2, where we plot $W_{SS}(A)$ for $\gamma_b=0$ at different values of the energy density, it can be seen how the system, which in this case is thermodynamically closed in an effective way, is always on the border of the ''limit cycle'' regime.

The role played by $u(2\omega_0)$ in fixing the parameters characteristic of the dynamics is rooted in the form chosen for $V(q)$. A study with coupling functions $V(q)\sim q^{\rho}$ $(\rho>2)$ can be carried out, the main difference with the previous description being the presence of parametric noise linked to nonlinear functions in the equation for A. Given that the particular form $A\zeta_{p,1}(t)$ of the stochastic term is determinant in the appearance of the on-off signatures, a qualitatively different behavior is therefore expected. We also point out that if nonlinear interaction between the phonons and the coupling of these to the thermal bath were considered, as it was done with a heuristic approach in Ref. [7], the resulting spectral changes in the colored noise could alter the efficiency of the initial state to generate the oscillations.

B. Time-dependent solutions

Time-dependent solutions of the Fokker-Planck equation for A can be obtained using for $W(A,t)$ an eigenfunction expansion with discrete and continuum branches [11]. For the off state the analysis is simpler: we can neglect the nonlinear terms and perform a derivation of the length distribution through a first-passage time study. In effect, with this approximation the time evolution of the amplitude is converted, through a proper change of variable, into a Wiener process, and an analytical expression for the time-dependent probability density is readily obtained; consequently, the first-passage time study can be straightforwardly performed. In this way we have found the two essential features of this phase: the power-law function with an exponential tail $P(s)=s^{-3/2}\exp(-s/s^*)$, where $s^*\sim\nu^{-2}$, and the dependence of the mean duration with the coupling strength $\bar{s}\sim\nu^{-1}$.

Approximate correlation functions can also be derived analytically: whereas for $\nu>2$ the dominance of the discrete part of the spectrum leads to exponential decay, in the *threshold region* $0\leqslant\nu\leqslant2$, which includes the regime in which intermittency sets in, the correlation function is completely determined by the continuous branch and, in the asymptotic limit and for large τ, is given by [11]

$$\lim_{t\to\infty}\langle A(t+\tau)A(t)\rangle-\langle A(t)\rangle^2\simeq\begin{cases}\dfrac{\Gamma^4\left(n+\dfrac{1}{2}\right)}{\pi n!\,\Gamma(2n)}e^{-2n^2Q\tau}\left[\left(\dfrac{\pi Q\tau}{2}\right)^{-1/2}+O(\tau^{-3/2})\right], & \nu=2n\\[2ex] \dfrac{\Gamma^2\left(-\dfrac{\nu}{2}\right)\Gamma^4\left(\dfrac{\nu+1}{2}\right)}{8\Gamma(\nu)}e^{-\frac{\nu^2Q\tau}{2}}\left[\left(\dfrac{\pi Q\tau}{2}\right)^{-3/2}+O(\tau^{-5/2})\right], & 0<\nu<2,\end{cases}$$

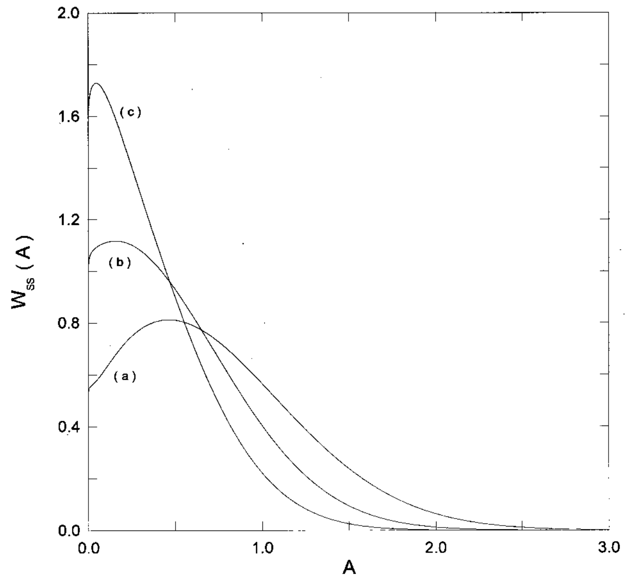

FIG. 3. Stationary probability density $W_{SS}(A)$ for a nonzero temperature ($k_B T=0.02$), $\gamma_b=0.01$, and $u(2\omega_0)=0.3$ (a), $u(2\omega_0)=0.2$ (b), and $u(2\omega_0)=0.143$ (c).

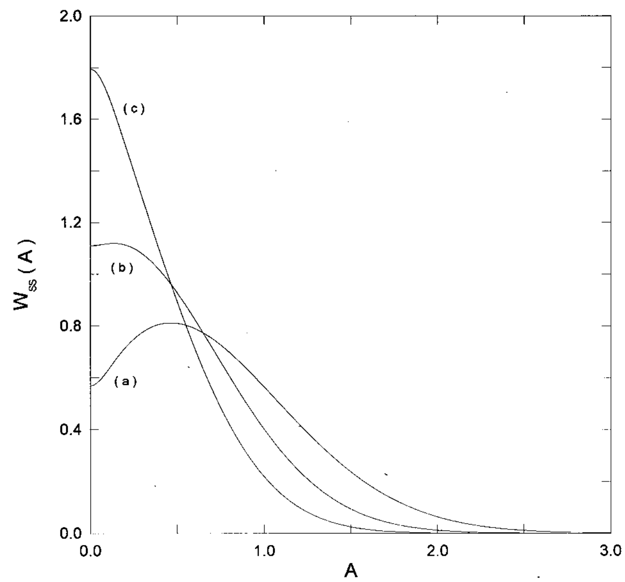

FIG. 4. Same as Fig. 3 for $\eta=0$.

where $Q=\alpha^2\gamma_p u(2\omega_0)/(2\omega_0^2)$. Hence, the system relaxes in a nonexponential way. The spectral density of the bidimensional system is determined by the time dependence of this correlation function and also by the additional fluctuations in the phase; in this sense we study now the role of the noise terms in the equation for φ. In the considered case of zero temperature, the mean frequency is obtained performing a statistical average in the expression for $\dot{\varphi}$ in Eq. (6) and taking into account the definition of ψ. In this way we have

$$\langle \dot{\psi} \rangle = \omega_0 - \frac{3\alpha^2\gamma_p\Omega_c}{8\pi\omega_0}\frac{\nu}{\Lambda} + m, \tag{8}$$

where the second term on the right-hand side has its origin in the nonlinear character of the potential [in our first-order perturbative treatment this is the only effect of nonlinearities in $\tilde{U}(q)$]; the third term is due to the finite correlation time of the multiplicative noise of Eq. (2). Obviously, these terms must be small compared with ω_0 for the applied methodology [coarse graining of Eq. (2)] to be valid. In the particular case in which the initial state of the phonons gives rise to a broadband noise $\xi_p(t)$ whose spectral density, centered on $2\omega_0$, with a sufficiently small correlation time $1/\lambda$, and a strength σ, is given by

$$S[\xi_p;\omega] = 8\gamma_p u(\omega) = 4\lambda\sigma^2 \frac{\omega^2+(2\omega_0)^2+\lambda^2}{[\omega^2-(2\omega_0)^2-\lambda^2]^2+4\lambda^2\omega^2}, \tag{9}$$

the additional shift m can be explicitly written as

$$m = -\frac{\alpha^2\sigma^2}{2\omega_0(\lambda^2+16\omega_0^2)} < 0. \tag{10}$$

The relevance that these shifts and the additive noise in the dynamics of the phase can have in the spectral characteristics of the whole process was qualitatively discussed in Refs. [13] and [14]. It was shown how these features can explain the widening of the output signal and the reduction of the peak frequency, found for increasing noise intensities in the spectra of systems similar to our reduced oscillator [12]. Given the more complex character of the present model, we cannot derive a compact expression for its spectrum. Nevertheless, the previous analysis gives us some clues to conjecture: first, the nonexponential decay of the correlation function for the amplitude in the regime in which intermittency appears must play the key role in determining the spectral features of the system; and second, the dynamics of the angular variable can be responsible for nontrivial changes in the spectra.

IV. THE EFFECT OF ADDITIVE NOISE ON ON-OFF INTERMITTENCY

Let us analyze how robust the previous description is against small increments of the temperature. For $T\neq 0$, the equation for A presents two additional terms: the additive noise $(1/\omega_0)\zeta_{b,1}$, typical of the standard models for noisy on-off intermittency [15,16], and the *deterministic* force $(\gamma_b kT)/(2\omega_0^2)A^{-1}$, absent in those models and specific to the present self-consistently reduced derivation. We can differentiate the effects of each of these terms: the additive noise makes the singularity at the origin disappear, altering the onset of intermittency and some of its signatures [11,15]; the *deterministic* term gives rise to a vanishing probability density at the origin and to a shift of the most probable amplitude to a temperature-dependent value. These effects increase with T and can be relevant inside the threshold region. The different functional form of the probability distribution, which now becomes

$$W_{SS}(A) = N\left(\gamma_b k_B T + \frac{\alpha^2\gamma_p u(2\omega_0)}{2}A^2\right)^{\nu-1/2+\beta\Lambda-\eta} \times A^{2\eta}e^{-\Lambda A^2}, \tag{11}$$

where N is a normalization constant, $\eta = \gamma_b k_B T/[\alpha^2 \gamma_p u(2\omega_0)]$, and $\beta = 2\eta$, accounts for these considerations. In Figs. 3 and 4 we depict, respectively, this probability density and its counterpart in a typical model for noisy on-off intermittency obtained by making $\eta = 0$ in Eq. (11). For a sufficiently small value of T, the presence of the *deterministic* term hardly introduces any relevant change: it is only in a very small region near the origin that the dynamics is altered. Therefore, for small T, we can consider that the laminar phase includes this region, and consequently we can still describe the switching as intermittent behavior.

V. CONCLUDING REMARKS

In conclusion, we have found that on-off intermittency can appear in a complex system coupled to a thermal reservoir as a result of the existence of a nonequilibrium state of ‘‘hidden modes’’ nonlinearly coupled to the mode observed. Since the parameters of the reduced equations have a well traced microscopic origin, some physical implications can be extracted from the properties of the emergent dynamics. First, the *coupling parameter* ν, on which the specific properties of the effect depend and which in standard models [2–6] corresponds to the coupling strength that incorporates the chaotic or stochastic signal into the *slaved* system, is in our model a measure of the difference between the assumed mechanism of dissipation and the one existent in a thermodynamically closed system. The nonlinear coupling to the phonons gives a self-sustained character to the reduced oscillator, whereas the dissipation to the thermal bath results in a nontrivial $\nu \neq 1$. Second, the threshold character is linked to the limited efficiency of the environment to sustain oscillations in the relaxation process. Third, as the signatures are determined by the linear terms (both deterministic and stochastic) present in the equation for the amplitude, the role played by the functional form of $V(q)$ in the appearance of the phenomenon is crucial. The possible relevance of the model in different contexts is supported by the frequent presence in real physical problems of effective nonlinear couplings and perturbations to thermal states.

ACKNOWLEDGMENT

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[1] For a characterization of on-off intermittency, see J.F. Heagy, N. Platt, and S.M. Hammel, Phys. Rev. E **49**, 1140 (1994), and references therein.

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